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Less naive about supersymmetric lattice quantum mechanics

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Abstract: We explain why naive discretization results that have appeared in [1] do not appear to yield the desired continuum limit. The fermion propagator on the lattice inevitably yields a diagram with nonvanishing UV degree $D = 0$ contribution in lattice perturbation theory, in contrast to what occurs in the continuum. This diagram gives a finite correction to the boson 2-point function that must be subtracted off in order to obtain the perturbation series of the continuum theory, in the limit where the lattice spacing a vanishes. Using a transfer matrix approach, we provide a nonperturbative proof that this counterterm suffices to yield the desired continuum limit. This analysis also allows us to improve the action to $\mathcal{O}(a)$. We demonstrate by Monte Carlo simulation that the spectrum of the continuum theory is well-approximated at finite but small a, for both weak and strong coupling. We contrast the above situation for the naive lattice action to what occurs for the supersymmetric lattice action, which preserves a discrete version of half the supersymmetry. There, cancellations between $D = 0$ diagrams occur, obviating the need for counterterms.

Keywords: Lattice Quantum Field Theory, Supersymmetric Effective Theories.

Contents

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1. Introduction

In this article we study lattice definitions of the simplest sort of 1d supersymmetric quantum mechanics (SQM).[1] The continuum theory is invariant under two independent supersymmetries (*susys*) whose algebra is $\{Q_1, Q_2\} = 2i\partial_t$. From the imaginary-time continuum action, one can easily write down a naive lattice action. The next task is to study the quantum continuum limit. This exercise proves instructive as it illustrates certain issues and subtleties in the continuum limit of a lattice theory that contains fermions. The lessons that are learned have relevance to the lattice definition of supersymmetric field theories, which are the prime motivation of this work.

The naive discretization breaks both of the supersymmetries of the continuum theory. A superior lattice action has been constructed by Catterall and Gregory (CG) [1]. Their action possesses one exact lattice susy that is a discrete version of, say, Q_1. However, for an interacting theory, this lattice action cannot be made simultaneously invariant with respect to a discrete version of Q_2. The susy lattice action also follows from a general superfield analysis [3] of SQM actions that preserve 1 nilpotent supercharge, as well as the topological field theory approach of [4]. CG have studied the susy lattice theory by Monte

[1] For a review, and extensive references, see [2].

Carlo simulation. Some results found by CG are of interest to us in this article. CG have computed the bosonic and fermion mass gaps. They find that the spectrum is degenerate, already at finite lattice spacing a. CG have compared these results to what occurs for the case of naive discretization. They find that the naive discretization gives a very different answer. Indeed, looking at their results, it would appear that the naive discretization does not yield the spectrum degeneracy of the continuum theory in the $a \to 0$ limit. In particular, the bosonic mass gap does not approach the correct value in this limit.

This result seems rather strange. CG have attributed it to large nonperturbative renormalizations. In this article we take a closer look at the issue. We find that the modified fermion propagator that occurs on the lattice gives a nonvanishing diagram with UV degree[2] $D = 0$. In such a diagram, modes at the UV cutoff a^{-1} do not decouple in the $a \to 0$ limit, and their effects need to be subtracted off with a counterterm. The susy lattice action is superior in this regard: an additional $D = 0$ diagram appears, and just cancels the $D = 0$ diagram of the naive theory. Consequently, UV modes do decouple in the $a \to 0$ limit, and there is no need to introduce counterterms. These features explain the discrepant results of CG.

We now summarize the remainder of this article. In section 2, we review relevant aspects of the continuum theory. Then we describe lattice actions and the effect of discretization on susy. In section 3, we discuss lattice perturbation theory and make our main points. The modifications to the UV behavior of perturbation theory that occur on the lattice have an important effect: one has to add a finite counterterm to ensure that a $D = 0$ graph cancels. The lattice theory then has only graphs of negative UV degree; Reisz's theorem [5] guarantees that they approach their continuum value, leading to the desired supersymmetric continuum theory. The absence of these counterterms in [1] explains why CG did not obtain a susy continuum limit from the naive lattice action. In Section 3 we also present a transfer matrix analysis, which provides a nonperturbative proof that the counterterm suffices to guarantee the continuum limit. Furthermore, this analysis allows us to derive the $\mathcal{O}(a)$-improved action.

In section 4, we describe Monte Carlo simulations of the naive lattice action, with and without corrections. We explore the effect of both (i) the necessary $\mathcal{O}(1)$ subtraction and (ii) the $\mathcal{O}(a)$ improvement that were derived in section 3. We find that these corrections are sufficient to yield good agreement with the continuum at small but nonzero a. In section 5 we draw our conclusions.

2. Preliminaries

In this section, we review elementary aspects of the continuum theory and set out our notations. Certain features of the naive lattice action and susy lattice action will be discussed. These considerations will set the stage for our later perturbative, transfer matrix, and Monte Carlo analyses of the quantum continuum limit.

[2]Our definition of UV degree is the same as in [5]. An explicit description of UV degree will be given below.

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2.1 The continuum theory

The hamiltonian of the theories that we study is of the form:

$$\begin{aligned} H_{SQM} &= \frac{1}{2}p^2 + \frac{1}{2}h'^2(q) - \frac{1}{2}h''(q)[b^\dagger, b]\,, \\ [q,p] &= i\,, \qquad \{b, b^\dagger\} = 1\,, \qquad b^2 = (b^\dagger)^2 = 0\,. \end{aligned} \tag{2.1}$$

Here, q and p are the bosonic coordinate and momentum operators, and $b^\dagger, b$ create and destroy the fermionic state. That is to say, a basis of states is $|x, \pm\rangle$ where

$$q|x,\pm\rangle = x|x,\pm\rangle\,, \qquad b|x,-\rangle = 0\,, \qquad b^\dagger|x,-\rangle = |x,+\rangle\,. \tag{2.2}$$

We will refer to the function $h(q)$ that appears in (2.1) as the superpotential; there, $h'(q) = \partial h(q)/\partial q$, etc. It is well-known that the theory is invariant under two independent susys whose algebra is $\{Q_1, Q_2\} = 2H_{SQM} = 2i\partial_t$. These are nothing but $Q_1 = b(ip + h'(q))$ and $Q_2 = b^\dagger(-ip + h'(q))$.

It is straightforward to write a imaginary-time action corresponding to (2.1):

$$S = \int_0^\beta dt \left[\frac{1}{2}(\dot{x}^2 + h'^2(x)) + \bar{\psi}(\partial_t + h''(x))\psi\right]. \tag{2.3}$$

Here, ψ and $\bar{\psi}$ are independent Grassmann fields, and x is real. All are one-component variables. The action is invariant under the imaginary-time continuation of the susy transformations, generated by infinitesmal Grassmann parameters ϵ_1, ϵ_2:

$$\delta x = \epsilon_1\psi + \epsilon_2\bar{\psi}\,, \qquad \delta\bar{\psi} = -\epsilon_1(\dot{x} + h')\,, \qquad \delta\psi = -\epsilon_2(\dot{x} - h')\,. \tag{2.4}$$

To prove this invariance one need only make use of periodic boundary conditions for the fields. Equations of motion are not involved, which is important since they do not generally have a solution in the imaginary-time theory. With $\delta = \epsilon_1 Q_1 + \epsilon_2 Q_2$,

$$\begin{aligned} Q_1 x &= \psi\,, \qquad Q_1\bar{\psi} = -(\dot{x} + h')\,, \qquad Q_1\psi = 0\,, \\ Q_2 x &= \bar{\psi}\,, \qquad Q_2\bar{\psi} = 0\,, \qquad Q_2\psi = -(\dot{x} - h')\,, \end{aligned} \tag{2.5}$$

defines the supercharges Q_1, Q_2.

In what follows it will be convenient to distinguish as $\tilde{h}$ the part of the superpotential that leads to interaction terms in the action:

$$h = \frac{1}{2}mx^2 + \tilde{h}\,, \qquad \tilde{h} = \sum_{n>2} \frac{g_n}{n}x^n\,. \tag{2.6}$$

A particularly simple case that we will concentrate on is the one studied by CG:

$$h = \frac{1}{2}mx^2 + \frac{1}{4}gx^4\,. \tag{2.7}$$

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2.2 The naive action

Using a naive discretization of the bosons, and Wilson fermions to avoid spectrum doublers, one obtains the following naive lattice action:

$$a^{-1}S = \frac{1}{2}\Delta^- x_i \Delta^- x_i + \frac{1}{2}h'_i h'_i + \bar{\psi}_i(\Delta^W(r)_{ij} + h''_i \delta_{ij})\psi_j \,. \tag{2.8}$$

(For convenience — here and below — we have moved the factor of a that comes from the dt of the continuum to the l.h.s.; repeated indices are summed.) Throughout this article, we denote the finite difference operators that we use according to:

$$\begin{aligned} \Delta^+_{ij} &= \frac{1}{a}(\delta_{i+1,j} - \delta_{ij})\,, \qquad \Delta^-_{ij} = \frac{1}{a}(\delta_{ij} - \delta_{i-1,j})\,, \\ \Delta^S &= \frac{1}{2}(\Delta^+ + \Delta^-)\,, \qquad \Delta^2 = \Delta^-\Delta^+ = \Delta^+\Delta^-\,, \end{aligned} \tag{2.9}$$

and $\Delta^W(r)$ is the Wilson operator, defined in (2.11) below. We could have chosen the forward difference operator Δ^+ just as well in the bosonic part of the action; simple identities show that for periodic boundary conditions

$$\Delta^- x_i \Delta^- x_i = \Delta^+ x_i \Delta^+ x_i \,. \tag{2.10}$$

As stated, we have introduced the Wilson operator

$$\Delta^W(r) = \Delta^S - \frac{ra}{2}\Delta^2 \tag{2.11}$$

in the fermion kinetic term. In 1d it interpolates between various choices for the finite difference approximation to the time derivative:

$$\Delta^W(r = \pm 1) = \Delta^\mp, \qquad \Delta_W(r = 0) = \Delta^S \,. \tag{2.12}$$

Thus, the naive choices Δ^+ or Δ^- for the fermion kinetic term are implicitly contained in (2.8). Taking advantage of (2.10), we obtain the naive discretization of all time derivatives appearing in (2.3) according to $\partial_t \to \Delta^-$ if $r = 1$ and $\partial_t \to \Delta^+$ if $r = -1$. These are free of doublers, whereas Δ^S has doublers. There are two reasons why we use the more general operator $\Delta^W(r)$.

First, for $d > 1$, i.e. in the field theories that we ultimately have in mind, the naive choices of $\partial_\mu \to \Delta^\pm_\mu$ ($\mu = 1, \ldots, d$) always lead to fermion doublers, as does Δ^S_μ. By contrast, the d-dimensional Wilson operator $\Delta^W_\mu(r) = \Delta^S_\mu - (ra/2)\Delta^2$ does not have doublers, provided $r \neq 0$; it does not interpolate between the naive discretizations in $d > 1$. (Here, $\Delta^2 = \sum_\nu \Delta^+_\nu \Delta^-_\nu$ is the d-dimensional lattice laplacian.) Thus, we seek to understand the role of the Wilson operator in the simpler 1d case as a foundation for $d > 1$ lattice susy studies.

Second, the Wilson operator for r such that $0 \ll |r| \ll 1$ gives a "physical" interpretation to UV modes, as states of mass $\mathcal{O}(ra^{-1})$. This is particularly useful for our purposes in section 3. We find a lack of decoupling of these "lifted doublers" in a diagram with UV degree $D = 0$ (cf. section 3.1). It is this effect that is responsible for a finite violation of susy Ward identities (cf. section 3.3) — necessitating a finite counterterm to the naive action. Thus, we use the Wilson operator to illustrate the origin of the corrections coming from the UV scale.

We choose to replace the continuum susy by a discrete approximation $\partial_t \to \Delta^+$:

$$
\begin{aligned}
Q_1 x_i &= \psi_i\,, \qquad Q_1\bar\psi_i = -(\Delta^+ x_i + h_i')\,, \qquad Q_1\psi_i = 0\,,\\
Q_2 x_i &= \bar\psi_i\,, \qquad Q_2\bar\psi_i = 0\,, \qquad Q_2\psi_i = -(\Delta^+ x_i - h_i')\,.
\end{aligned} \tag{2.13}
$$

(Other choices could be made; they do not significantly alter our arguments. For example, it is possible to remove the quadratic terms in (2.14) below. However, it can be shown that the remaining terms will still lead to a finite violation of the susy Ward identities.) It is then straightforward to work out the variation of the action. For example,

$$
\begin{aligned}
a^{-1}Q_1 S &= -\frac{a}{2}(1+r)x_i\Delta^-\Delta^2\psi_i - \frac{a}{2}(1-r)m x_i\Delta^2\psi_i + \frac{ra}{2}\tilde h_i'\Delta^2\psi_i +\\
&\quad + (\Delta^S\tilde h_i' - \tilde h_i''\Delta^+ x_i)\psi_i\,.
\end{aligned} \tag{2.14}
$$

In each equation we have substituted (2.6). Now we note that (no sum over i implied):

$$
\Delta^S\tilde h_i' - \tilde h_i''\Delta^+ x_i = -\frac{a}{2}\tilde h_i''\Delta^2 x_i + \mathcal{O}(a^2)\,. \tag{2.15}
$$

Thus we obtain that all terms on the r.h.s. of (2.14) are suppressed by a factor of a. (As is well known, this $\mathcal{O}(a)$ violation is due to the failure of the Leibnitz rule on the lattice [6].) A similar statement holds for $Q_2 S$. The classical continuum limit therefore yields

$$
Q_A S = a\mathcal{Y}_A \equiv a\sum_i aY_{A,i} \to \int_0^\beta dt\; aY_A(t) \to 0\,, \qquad A = 1,2\,. \tag{2.16}
$$

The susy invariance is recovered in this limit. On the other hand, in section 3 we will find that in the quantum theory there is a finite violation of the continuum susy Ward identities:

$$
\langle Q_A S\rangle = a\langle\mathcal{Y}_A\rangle \sim a\cdot a^{-1} \to \text{finite}\,. \tag{2.17}
$$

The main point of this article is to explain why this occurs and how to cure it.

2.3 The susy lattice action

Here the action is the one that appeared first in [1]. It preserves a discrete version of half the continuum susy exactly. The action was subsequently rederived in [3] using a lattice superfield approach, and in [4] using a topological field theory approach. It is given by (we suppress the r-dependence of Δ^W):

$$
a^{-1}S = \frac{1}{2}\sum_i(\Delta^W x_i + h_i')^2 + \sum_{ij}\bar\psi_i(\Delta^W_{ij} + h_i''\delta_{ij})\psi_j\,. \tag{2.18}
$$

Comparing to the naive action (2.8), it can be seen that the fermionic part of the action is unchanged. By contrast, the bosonic part of the action differs from the naive discretization (2.8) in a number of ways. Due to supersymmetrization, the Wilson operator

Δ^W appears in both the boson and fermion kinetic terms. It follows that at $r = 0$ the bosons also have doublers; these are similarly lifted by a Wilson mass term $m_W = m - (ra/2)\Delta^2$ for $r \neq 0$. This degeneracy of UV modes is necessitated by the one exact lattice susy. The lattice susy also requires derivative interaction terms that are not present in the continuum theory. To see this, note that the bosonic part of the action S_B may be written as:

$$\begin{aligned} a^{-1}S_B = & \frac{1}{2}x_i\left[-\Delta^S\Delta^S + \left(m - \frac{ra}{2}\Delta^2\right)^2\right]x_i + m\tilde{h}'_i x_i + \frac{1}{2}\tilde{h}'_i\tilde{h}'_i + \\ & + \tilde{h}'_i\Delta^S x_i - \frac{ra}{2}\tilde{h}'_i\Delta^2 x_i\,. \end{aligned} \tag{2.19}$$

It is clear from (2.19) that in addition to the modified quadratic action, there are also (in the second line) two sorts of interaction vertices not present in the continuum. The vertices associated with $\tilde{h}'_i\Delta^S x_i$ come from the $h'_i\Delta^W x_i$ crossterm one gets from the square in (2.18). As pointed out in [3], these interactions break reflection positivity; nevertheless the continuum hamiltonian and Hilbert space are recovered after a "conjugation" of states and the transfer matrix — by the operators $\exp(\pm h(q))$, where $q|x_i\rangle = x_i|x_i\rangle$. It should also be noted that whereas $\int \tilde{h}'\partial x = \int \partial\tilde{h}$ vanishes in the continuum with periodic boundary conditions, on the lattice $\tilde{h}'_i\Delta^S x_i$ is not purely a total derivative, due to the failure of the Leibnitz rule. The vertices associated with $ra\tilde{h}'_i\Delta^2 x_i$ are "Wilson interaction terms." Both of these sets of vertices have UV degree $D = 1$, so that lattice power counting differs from that of the continuum. We will examine this issue in section 3.4 below.

As mentioned above, our detailed considerations will focus on the superpotential (2.7). In what follows, we find it convenient to set $r = 1$; this leads to various simplifications of the above formulae, which we summarize here:

$$\begin{aligned} M_{ik} &\equiv a(\Delta^-_{ik} + h''_{ik}) = -\delta_{i-1,k} + \mu_i\delta_{ik}\,, \\ \mu_i &= 1 + a(m + 3gx_i^2)\,, \qquad \det M = -1 + \prod_i \mu_i\,. \end{aligned} \tag{2.20}$$

The matrix M is the one which appears in the fermionic part of both the naive and susy actions: $S_F = \bar{\psi}_i M_{ik}\psi$. For $m > 0$ and $g > 0$, the fermion determinant is strictly positive. This is a happy state of affairs, given the sign/complex-phase problems that are often experienced in susy lattice systems; for example [7]. We note, however, that when h is a polynomial of odd degree, positivity will not hold. (This happens to be the case where it is known from the continuum that susy is spontaneously broken.)

qAs first pointed out by CG, the action may be written in the *Nicolai map* form:

$$a^{-1}S = \frac{1}{2}\mathcal{N}_i\mathcal{N}_i + \bar{\psi}_i\frac{\partial\mathcal{N}_i}{\partial x_k}\psi_k\,, \qquad \mathcal{N}_i = \Delta^W x_i + h'_i\,. \tag{2.21}$$

This form makes the exact susy rather obvious:

$$\delta_1 x_i = \epsilon_1\psi_i\,, \qquad \delta_1\psi_i = 0\,, \qquad \delta_1\bar{\psi}_i = -\epsilon_1\mathcal{N}_i\,. \tag{2.22}$$

The index "1" indicates the correspondence to the ϵ_1 part of the continuum susy (2.5), the lattice version of Q_1 which has been preserved exactly on the lattice. It is easy check that this is also a symmetry of the partition function measure. This comes with the important qualification that periodic boundary conditions are always assumed for both bosons and fermions. Thus our partition function is $Z = \mathrm{Tr}(-1)^F e^{-\beta H}$, the Witten index. Continuum (imaginary-time) vacuum expection values are obtained in the limit $\beta \to \infty$.

3. Counterterm analysis

We will show below (section 3.1) that the naive lattice action does not give rise to a finite lattice perturbation series, in the technical sense that a UV degree $D = 0$ proper vertex exists. This is in contrast to the continuum ($D = 0$ contributions vanish) and the susy lattice ($D = 0$ contributions cancel — see section 3.4). A local counterterm can be introduced to render the naive lattice theory finite.[3] We determine this counterterm and show by various means that the subtracted theory has the correct continuum limit. In the process, we are able to compute the $\mathcal{O}(a)$-improved action (cf. section 3.2), which has a faster convergence toward the continuum limit. In section 3.3, we illustrate how the continuum Ward identities are recovered in the $a \to 0$ limit, as a consequence of the finite counterterm.

The $D = 0$ diagrams that occur are just 1-loop corrections to the boson propagator. $D \geq 0$ primitive diagrams do not occur at higher loops. We will find that the $a \to 0$ results in the naive lattice case differ from that of the continuum due to modes at the edges of the Brillouin zone. Thus the discrepancy can be associated with fermion doublers that are lifted by the Wilson mass — the contribution of the second term in (2.11) at the edges of the Brillouin zone. This non-negligible effect of doublers should not come as a surprise: the effects of lifted doublers, with $\mathcal{O}(a^{-1})$ mass, need not decouple in a $D = 0$ diagram. Rather, we find that they contribute a finite result. (While a $D = 0$ diagram could contribute a term $\sim \ln a$, we will find that this is not the case and only an a-independent counterterm will be required.)

As will be shown in section 3.4, in the susy lattice action, an additional $D = 0$ diagram appears. It cancels the finite correction that appears in the naive lattice theory, alleviating the need for a counterterm. The additional diagram arises because of the boson interactions associated with supersymmetrization of the Wilson mass term. Thus one advantage of the susy lattice action is that it provides a lattice perturbation theory that is finite (in the technical sense — only $D < 0$ contributions survive for all proper vertices). By the theorem of Reisz [5], the susy lattice action therefore requires no counterterms to achieve the desired continuum limit. This explains why it was previously found [1] that the susy lattice action extrapolated well to the continuum, whereas the naive lattice action did not.

We then show in section 3.2 that the counterterm that is required for the naive action can be seen very easily in the transfer matrix description utilized in [3]. Remarkably, we are able to determine the counterterm that is required for an arbitrary superpotential, without having to perform any loop calculations. With a little more algebra, we are able to obtain the $\mathcal{O}(a)$-improved action. The transfer matrix analysis has the advantage that it is

[3] The spirit of this analysis is quite similar to that of [8], where a super-renomalizable supersymmetric field theory was obtained from a lattice theory by the treatment of counterterms.

nonperturbative. Thus we are able to conclude that the counterterm suffices to guarantee the continuum limit beyond perturbation theory. This is consistent with our simulation results, which will be discussed in section 4.

3.1 Power counting for the naive lattice theory

As elsewhere, our focus here will be on the case of the superpotential (2.7). We will include all quadratic terms, including the Wilson mass for the fermion, in the free propagator. This does not alter the UV degree of internal fermion lines: since $(ra/2)\Delta^2 \sim a^{-1}$ at the edges of the Brillouin zone, the Wilson fermion propagator has the same degree[4] $D = 1$ as $\Delta^S \sim a^{-1}$. The interaction vertices can be worked out from (2.8):

$$a^{-1} S_{\text{int}}^{\text{naive}} = \sum_i \left(mgx_i^4 + \frac{1}{2} g^2 x_i^6 + 3gx_i^2 \bar{\psi}_i \psi_i \right) . \tag{3.1}$$

We count the number of x^4 vertices by V_4, the number of x^6 vertices by V_6, and the number of $x^2 \bar{\psi}\psi$ vertices by $\tilde{V}_4$. Each of these vertices has UV degree $D = 0$. Then in the usual manner a diagram will satisfy the following relations between the UV degree D, the number of loops L, the number of internal boson and fermion lines I_B, I_F resp., the number of external boson and fermion lines E_B, E_F resp., and the various vertices:

$$\begin{aligned} D &= L - 2I_B - I_F \,, \qquad L = 1 + I_B + I_F - V_4 - \tilde{V}_4 - V_6 \,, \\ E_B + 2I_B &= 4V_4 + 6V_6 + 2\tilde{V}_4 \,, \qquad E_F + 2I_F = 2\tilde{V}_4 \,. \end{aligned} \tag{3.2}$$

The only $D \geq 0$ solution with $L > 0$ is:

$$D = E_F = I_B = V_4 = V_6 = 0 \,, \qquad L = I_F = \tilde{V}_4 = 1 \,, \qquad E_B = 2 \,, \tag{3.3}$$

the 1-fermion-loop correction to the boson 2-point function, diagram (a) in figure 1. In the continuum, the corresponding expression is:

$$\Sigma_{cont.} = 6g \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \frac{-ip + m}{p^2 + m^2} = 6g \left(\frac{1}{2} + \mathcal{O}(ma) \right) . \tag{3.4}$$

(Note that we take the UV cutoff to be $\Lambda = \pi a^{-1}$, as is usual when relating lattice perturbation theory to the continuum.) We notice that the $D = 0$ part of the diagram (i.e., involving the $-ip$ part of the numerator) vanishes because it is odd w.r.t. $p \to -p$. For this reason, the result is finite in the $a \to 0$ limit, rather than log-divergent as we would

[4] The lattice degree of divergence of a lattice Feynman diagram is defined as the exponent D in the scaling a^{-D} of a lattice Feynman graph at small lattice spacing. In the present case of $d = 1$, a factor a^{-1} is assigned to every loop momentum integral (finite sum). The small-a scaling of bosonic (a^2) and fermionic (a^1) propagators is read off the expressions for lattice propagators. The scaling of vertices is read off from the vertex function that appears in the Feynman rule. For example, none of Feynman rules for the interactions in (3.1) scale with a; thus they have degree $D = 0$. On the other hand, the two new interaction in (3.22) below have Feyman rules $\sim ga^{-1}\sin(2\pi k/N) \sim a^{-1}$ and $\sim rag \cdot a^{-2}\sin^2(\pi k/N) \sim a^{-1}$ resp. Thus they have degree $D = 1$.

Figure 1: Diagrams contributing at $\mathcal{O}(g)$ to the boson propagator. Diagram (a) has only a $D = -1$ piece surviving in the continuum, but on the lattice has a $D = 0$ contribution coming from fermion doublers. In the susy lattice action, additional interactions due to supersymmetrization of the Wilson mass term cause diagram (b) to also acquire a $D = 0$ contribution, which just cancels that of diagram (a).

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naively expect based on power-counting alone. At the same order in perturbation theory, a 1-scalar-loop diagram (b) of figure 1 must be added to this; it differs only by a factor of -2 relative to (3.4), yielding the net result $-3g(1 + \mathcal{O}(ma))$.

On the lattice, this continuum expression is replaced by:

$$\Sigma_{latt.} = \frac{6g}{Na} \sum_{k=0}^{N-1} \frac{-ia^{-1}\sin(2\pi k/N) + m + 2ra^{-1}\sin^2(\pi k/N)}{a^{-2}\sin^2(2\pi k/N) + (m + 2ra^{-1}\sin^2(\pi k/N))^2} \,. \tag{3.5}$$

A naive $a \to 0$ limit of this expression would (i) set $p = 2\pi k/(aN)$, (ii) replace $(1/Na)\sum_k \to \int(dp/2\pi)$, and (iii) take the $a \to 0$ limit, with the physical momentum $p = 2\pi k/(aN)$ fixed, on the summand that appears in (3.5). Then one of course recovers the continuum fermion loop result (3.4). However, if the $a \to 0$ limit is taken after computing the sum, one finds that the fermion loop contribution to the boson 2-point function is twice as big as (3.4):

$$\Sigma_{latt.} = 6g\,(1 + \mathcal{O}(rma)) \,. \tag{3.6}$$

This result can be understood as follows. The $-ia^{-1}\sin(2\pi k/N) \sim -ip$ term is odd about $k = 0$ and vanishes under the sum, just as in the continuum. However, the Wilson mass term $2ra^{-1}\sin^2(\pi k/N)$ is even and does not. Using $p = 2\pi k/(aN)$, it can be written as $2ra^{-1}\sin^2(pa/2)$. In step (iii) of the naive limit described above, we would neglect this term. However, in the explicit evaluation of the sum (3.5), the Wilson mass term becomes $\mathcal{O}(a^{-1})$ at the edges of the Brillouin zone, where $k \approx N/2$. In order to take this non-negligible contribution into account, step (iii) should be avoided,[5] and the $a \to 0$ limit must be taken after computing the sum. Since the Wilson mass term scales like a^{-1}, the naive expectations based on power counting are correct in the lattice theory — unlike in the continuum theory — there is a regulator dependent contribution that must be subtracted off. (However, a log-divergent result is not obtained for this particular $D = 0$ diagram.) Whereas in the continuum we obtain (3.4), on

[5]Or, as we show below, one can be more careful with the region where $p = \mathcal{O}(a^{-1})$.

the lattice there is an extra (finite) contribution of 1/2 coming from the edges of the Brillouin zone. The doublers do not decouple, precisely because this is a $D = 0$ diagram.

In order to match the continuum theory, we must add a boson mass counterterm that compensates for the effects of the doublers in the above diagram:

$$\frac{6g}{Na}\sum_{k=0}^{N-1}\frac{2ra^{-1}\sin^2(\pi k/N)}{a^{-2}\sin^2(2\pi k/N)+(m+2ra^{-1}\sin^2(\pi k/N))^2}\to 3g\equiv\delta m^2\,. \tag{3.7}$$

The shifted action is then

$$S_c = S+\frac{a}{2}\sum_i\left[3gx_i^2+\text{const.}\right] = S+\frac{a}{2}\sum_i h_i''\,, \tag{3.8}$$

where we have added an overall constant $(m/2)$ to obtain the appealing form of the far r.h.s. It will turn out that this constant appears automatically in our transfer matrix analysis of section 3.2 below. We have only shown that a mass counterterm is required for the special case of h studied here, eq. (2.7). The expression (3.8) would, for a more general h, imply interaction counterterms as well. In section 3.2 we will show that the counterterm $h''/2$ is the correct choice in the general case.

We now give an intuitive understanding of why a finite counterterm is needed. Our discussion will also give the more careful treatment of the region where $p = \mathcal{O}(a^{-1})$, mentioned in Footnote 5, and will exploit the flexibility allowed by our use of the Wilson operator (2.11). Consider the case of r and a such that $r \ll 1$ but $m \ll ra^{-1}$. Then the sum (3.5) is well approximated as follows. The mass of the (lifted) doubler mode is just $M = 2ra^{-1}+m$, as can be seen by a change of variables to $p' = p-\pi a^{-1}$, and looking near small p'. Thus for $M \ll a^{-1}$, the doubler contributes like a particle of mass $M \gg m$. It follows that in this limit a good approximation to the integral that appears in (3.6) is just

$$\int_{-\pi/a}^{\pi/a}\frac{dp}{2\pi}\frac{m}{p^2+m^2}+\int_{-\pi/a}^{\pi/a}\frac{dp'}{2\pi}\frac{M}{p'^2+M^2} = \frac{1}{\pi}\left[\tan^{-1}\frac{\pi}{2am}+\tan^{-1}\frac{\pi}{2aM}\right]. \tag{3.9}$$

To $\mathcal{O}(r)$, each term on the r.h.s. of (3.9) yields 1/2. Thus as the UV cutoff a^{-1} is sent to infinity, the contribution of the doublers does not decouple. For $D \geq 0$ diagrams this lack of decoupling is usually associated with a divergence. Here, the lack of decoupling in a $D = 0$ diagram is instead only associated with a finite, a-independent contribution coming from the regulator.

Finally, we note that our result is consistent with the power-counting theorem of Reisz [5]. In essence, his theorem states that the naive continuum limit can be taken on the integrand of diagrams — step (iii) of the naive limit mentioned above — provided the UV degree of the diagram satisfies $D < 0$. That is, for $D < 0$ diagrams, the $a \to 0$ limit and the loop momenta integrals commute. However, this theorem does not apply to a $D = 0$ diagram, regardless of whether or not it is finite (in the sense of being a-independent versus $\ln a$ dependent).

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3.2 Transfer matrix analysis of the naive and improved lattice actions

The partition function Z_{naive} of the naive lattice action (2.8), with $r = 1$, can be represented as the trace of the N-th power of the transfer matrix operator, denoted here by $T[0,0]$, over the Hilbert space (2.2), weighted by $(-1)^F$ to account for periodic boundary conditions of fermions.[6] Thus, as we show in some detail in the appendix:

$$Z_{naive} = \mathrm{Tr}(-1)^F \; T[0,0]^N \, , \tag{3.10}$$

where $T[0,0] = T[k = 0, \ell = 0]$ can be read off eq. (3.12) below. In anticipation of counterterms, we generalize the naive action by functions $k(x)$ and $\ell(x)$:

$$h'^2(x) \to h'^2(x) + k(x) \, , \qquad h''(x)\bar{\psi}\psi \to (h''(x) + \ell(x))\bar{\psi}\psi \, , \tag{3.11}$$

and denote the corresponding "deformed" transfer matrix as $T[k,\ell]$, given by the following operator:

$$T[k,\ell] = \mathcal{N}(a) \int\limits_{-\infty}^{\infty} dz \exp\left[-\frac{z^2}{2a} - \frac{a}{2}h'^2(q) - \frac{a}{2}k(q)\right] e^{izp}\left[1 + a(h''(q) + \ell(q))b^\dagger b\right] \, , \tag{3.12}$$

where $\mathcal{N}(a)$ is an inessential a-dependent normalization constant. (In the appendix we show that the naive lattice partition function supplemented by the counterterms (3.11) can be written as $\mathrm{Tr}(-1)^F \; T[k,\ell]^N$.)

For small lattice spacing, one can use a saddle point approximation to evaluate the z-integral in (3.12). Then, it follows that the operator $T[k,\ell]$ of (3.12) can, for small a, be written as:

$$T[k,\ell] = \; e^{af(q)} \; e^{-aH[k,\ell]} \; e^{-af(q)} \, , \tag{3.13}$$

where f is given by:

$$f(q) = -\frac{1}{4}(h'^2(q) + k(q)) - \frac{1}{2}(h''(q) + \ell(q))b^\dagger b \, , \tag{3.14}$$

and the hamiltonian $H[k,\ell]$ is:

$$\begin{aligned} H[k,\ell] \; = \; & \frac{1}{2}p^2 + \frac{1}{2}h(q)'^2 - (h''(q) + \ell(q))\left(1 - \frac{1}{2}a(h''(q) + \ell(q))\right)\frac{1}{2}[b^\dagger, b] + \\ & + \frac{1}{2}k(q) - \frac{1}{2}(h''(q) + \ell(q))\left(1 - \frac{1}{2}a(h''(q) + \ell(q))\right) + \mathcal{O}(a^2) \, . \end{aligned} \tag{3.15}$$

Two important observations immediately follow from (3.13)–(3.15):

1. The $T[k,\ell]$ operator at small lattice spacing is equivalent, upon conjugation by e^{af}, to $\exp(-aH[k,\ell])$. When inserted in the trace, the factors $e^{\pm af}$ in (3.13) do not affect the partition function. (Actually, this conjugation can be avoided if one writes $T[k,\ell]$ in the standard form $e^{-\frac{a}{2}V(q,b^\dagger b)}e^{-\frac{a}{2}p^2}e^{-\frac{a}{2}V(q,b^\dagger b)}$, with an appropriate choice of V.) Thus, the general partition function $Z[k,\ell] \equiv \mathrm{Tr}(-1)^F T[k,\ell]^N$ approaches $\mathrm{Tr}(-)^F e^{-\beta H[k,\ell]}$ in the small-a, β-fixed limit, where $\beta \equiv Na$.

[6] The case of $r \neq 1$ can also be treated, but it is more complicated because the transfer matrix must have row and column indices labeled by pairs of sites.

2. The hamiltonian $H[0,0]$ — corresponding to the naive lattice action (2.8) — does not, even as $a \to 0$, approach the hamiltonian H_{SQM} (2.1) of SQM. Rather, as eq. (3.15) shows, $H[0,0] = H_{SQM} - \frac{1}{2}h''(q) + \mathcal{O}(a)$.

The second observation above shows that in order for the continuum limit of the naive lattice partition function to correspond to the path integral of SQM, the naive lattice action has to be supplemented by a finite counterterm: one must take $k(q) = h''(q)$ and $l(q) = 0$ in the transfer matrix (up to $\mathcal{O}(a)$ terms). This counterterm, with h of eq. (2.7), is precisely equal to the counterterm calculated in perturbation theory in the previous section, eq. (3.8).

Moreover, eq. (3.15) allows us to go further and demand that the correspondence to H_{SQM} is not corrected also at $\mathcal{O}(a)$. One chooses k and ℓ such that $H[k,\ell] = H_{SQM} + \mathcal{O}(a^2)$, which implies

$$\begin{aligned} k - (h'' + \ell)\left(1 - \frac{1}{2}a(h'' + \ell)\right) &= \mathcal{O}(a^2)\,, \\ h'' - (h'' + \ell)\left(1 - \frac{1}{2}a(h'' + \ell)\right) &= \mathcal{O}(a^2)\,. \end{aligned} \tag{3.16}$$

Thus, the $\mathcal{O}(a)$-improved naive lattice action should have a transfer matrix $T[k,\ell]$ with

$$k(q) = h''(q) + \mathcal{O}(a^2)\,, \qquad \ell(q) = \frac{1}{2}ah''^2(q) + \mathcal{O}(a^2)\,. \tag{3.17}$$

The $\mathcal{O}(a)$-improved action is therefore (recall we set $r = 1$):

$$a^{-1}S_{ca} = \frac{1}{2}\sum_i \left(\Delta^- x_i \Delta^- x_i + h_i' h_i' + h_i''\right) + \sum_{ij} \bar{\psi}_i \left(\Delta_{ij}^- + h_i''\delta_{ij} + \frac{1}{2}ah_i''^2\delta_{ij}\right)\psi_j\,. \tag{3.18}$$

The above analysis provides a complete, nonperturbative proof that the $\mathcal{O}(1)$ correction (3.8) to the bosonic part of the action suffices to guarantee the desired continuum limit. The $\mathcal{O}(a)$ improvement, which only corrects the fermionic part of the action, will be shown in our simulation results of section 4 to yield very good results in the $a \to 0$ limit.

3.3 Susy Ward identities in the naive lattice theory

It is easy to show that the following Ward identities hold on the lattice:

$$\langle Q_A \mathcal{O} \rangle = \langle (Q_A S)\mathcal{O} \rangle = a\langle \mathcal{Y}_A \mathcal{O} \rangle\,, \qquad A = 1,2\,. \tag{3.19}$$

Here $\mathcal{Y}_A$ are the lattice operators defined by (2.16), whose explicit form can be obtained from (2.14) and a similar expression for $Q_2 S$. A violation of the continuum Ward identity $\langle Q_1 \mathcal{O} \rangle = 0$ can only arise if the difference operators in $\mathcal{Y}_1$ generate a UV-divergent result, $\langle \mathcal{Y}_A \mathcal{O} \rangle = \mathcal{O}(a^{-n})$, $n > 0$. We now show that this indeed occurs in the uncorrected theory. This is in spite of the fact that the lattice perturbation series has only a $D = 0$ diagram, as shown in section 3.1. The point is that, as can be seen from (2.14), $\mathcal{Y}_1$ has vertices that have positive UV degree, whereas those that occur in the action (3.1) have vanishing UV degree. We will also illustrate how this difficulty is cured by the counterterm (3.8).

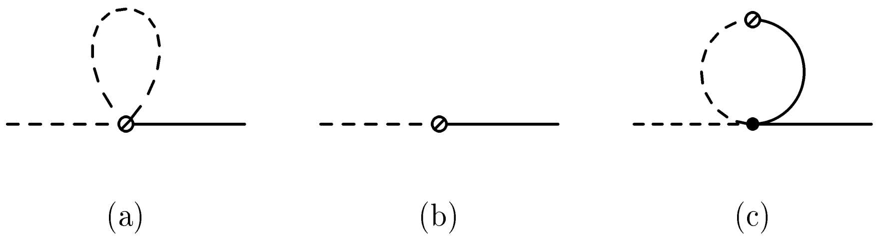

Figure 2: Diagrams associated with the cancellation of the $\mathcal{O}(g)$ violation of the continuum susy Ward identity. The 2-point and 4-point shaded vertices that violate fermion number are those arising from $Q_1 S_c$, where S_c is the corrected action indicated by (3.8). The sum of these diagrams vanishes in the $a \to 0$ limit, provided the external momentum satisfies $|p_{ext}| \ll a^{-1}$.

We note from (2.14) that each term in $\mathcal{Y}_1$ contains finite difference operators. When these operators act on would-be fermion doubler modes (i.e., doublers when $r = 0$), they give positive UV degrees. We will study the anomalous Ward identity associated with the operator $x_i \bar{\psi}_j$, since it is directly related to the lack of spectrum degeneracy in the unsubtracted naive case. Actually, we must be careful to restrict to the physical modes that are relevant to the "long-distance" physics. Operators at fixed time could easily be obtained by blocking the lattice fields, but it is simpler to just analyze the Ward identies in momentum space; i.e., using the Fourier transformed variables $\tilde{x}_k, \tilde{\bar{\psi}}_k$. Thus we will consider the Ward identities associated with the operator $\mathcal{O}_k = \tilde{x}_k \tilde{\bar{\psi}}_k$ subject to $k \ll N/2 \bmod N$. We will find that whereas $\langle Q_1 \mathcal{O}_k \rangle$ does not vanish in the continuum limit for the naive action, once the counterterm in introduced, the finite violation is subtracted off and the Ward identity is recovered in the $a \to 0$ limit.

Rather than study this cancellation directly, we will exploit the r.h.s. of the lattice Ward identity (3.19). We break $Q_1 S$ into operators that generate 2-point and 4-point vertices:

$$Q_1 S = (Q_1 S)_{(2)} + (Q_1 S)_{(4)} \equiv a\mathcal{Y}_{1,(2)} + a\mathcal{Y}_{1,(4)} \,. \tag{3.20}$$

The operator $\mathcal{Y}_{1,(2)}$ gives a $D = 3$ vertex and $\mathcal{Y}_{1,(4)}$ gives a $D = 2$ vertex. The leading order diagram associated with the latter vertex is shown in figure 2(a). The leading $D = 2$ piece vanishes. The surviving $D = 1$ part yields an $\mathcal{O}(a^{-1})$ result that just cancels the factor of a in (3.20), leading to a finite result of $3g(1 + \mathcal{O}(ma))$. The 2-point vertex also contributes at $\mathcal{O}(g)$, through the diagram of figure 2(c). Here, the $3gx^2\bar{\psi}\psi$ vertex of the action (3.1) is involved. The diagram has UV degree $D = 1$, again canceling the a in (3.20), yielding a finite result of $-6g(1 + \mathcal{O}(ma))$. The net result is finite and nonzero, which is consistent with the lack of spectrum degeneracy observed in [1]. The continuum Ward identity $\langle Q_1 \mathcal{O}_k \rangle = 0$ is not recovered in the $a \to 0$ limit, for the uncorrected naive action (2.8).

However, once the counterterm $(3/2)gx^2$ is added to the action,

$$Q_1 S \to Q_1 S + a \sum_i 3g x_i \psi_i \,. \tag{3.21}$$

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This contributes a 2-point $D = 0$ vertex that is not suppressed by a, and is indicated by the tree-level diagram in Fig. 2(b). This of course yields $3g$, canceling the finite violation that arises from the other two diagrams. Thus in the $a \to 0$ limit, we obtain to $\mathcal{O}(g)$ the continuum result $\langle Q_1 \mathcal{O}_k \rangle = 0$. We have verified that similar cancellations occur at $\mathcal{O}(g^2)$.

Of course, it is guaranteed that the cancellations will occur to all orders in perturbation theory, since the counterterm has ensured that we will recover the continuum perturbation series, which satisfies the susy Ward identities to all orders — for any operator $\mathcal{O}$, not just the one that we have considered here. Furthermore, our transfer matrix analysis provides a nonperturbative proof that the continuum hamiltonian is obtained in the quantum continuum limit. We necessarily recover the continuum result $\langle Q_A \mathcal{O} \rangle = 0$ in the corrected lattice theory when $a \to 0$.

3.4 Power counting for the susy lattice theory

In the transfer matrix analysis of [3], it was already proven for the susy lattice action (2.18) that the correct continuum limit is obtained without the need for counterterms. However, it is interesting to see the detailed cancellations that bring this about in perturbation theory.

In the special case (2.7), the interaction part of the action is given by:

$$a^{-1} S_{\text{int}}^{\text{susy}} = \sum_i \left(mgx_i^4 + \frac{1}{2} g^2 x_i^6 + gx_i^3 \Delta^S x_i - \frac{rag}{2} x_i^3 \Delta^2 x_i + 3gx_i^2 \bar{\psi}_i \psi_i \right). \tag{3.22}$$

Two new terms appear in (3.22), relative to the naive lattice theory (3.1). Both of these vertices scale like a^{-1} and thus contribute to D. We include these into the definition of the 4-point boson vertex, associated with V_4 in our power counting analysis. We then obtain a slightly modified version of (3.2):

$$\begin{aligned} D &= L + V_4 - 2I_B - I_F\,, \qquad & L &= 1 + I_B + I_F - V_4 - \tilde{V}_4 - V_6\,, \\ E_B + 2I_B &= 4V_4 + 6V_6 + 2\tilde{V}_4\,, \qquad & E_F + 2I_F &= 2\tilde{V}_4\,. \end{aligned} \tag{3.23}$$

It is straightforward to show that the only solutions with $D \geq 0$, $L > 0$ are just (3.3) and

$$D = E_F = I_F = \tilde{V}_4 = V_6 = 0\,, \quad L = I_B = V_4 = 1\,, \qquad E_B = 2\,, \tag{3.24}$$

corresponding to the scalar loop figure $1(b)$. Thus in the susy theory we have two $D = 0$ 1-loop diagrams to sum in the proper vertex associated with the boson 2-point function. It suffices to check at vanishing external momentum p_{ext}, as higher orders in the expansion about $p_{ext} = 0$ will have $D < 0$. One obtains from the boson loop diagram

$$-\frac{6g}{Na} \sum_{k=0}^{N-1} \frac{2m + 2ra^{-1} \sin^2(\pi k/N)}{a^{-2} \sin^2(2\pi k/N) + (m + 2ra^{-1} \sin^2(\pi k/N))^2}\,. \tag{3.25}$$

Added to the fermion loop diagram (3.5), the $D = 0$ contributions cancel. This is a consequence of the 1 exact susy of the lattice action. What is left over is a $D = -1$ result. By the theorem of Reisz [5], as $a \to 0$ this is guaranteed to give the continuum result — a finite mass correction $-3g$. (In the present case, it is trivial to check this explicitly.)

4. Simulation results

All of our simulation results correspond to the special case of superpotential (2.7) and $r = 1$. We will take $m = 10$, for the following reason. The effects of interactions always cause the mass gap m_1 (energy of the first excitation) to satisfy $m_1 > m$. Thus the Compton wavelength of this mode is always shorter than 1/10. For this reason we can safely set the system size $\beta = Na$ to $\beta = 1$: the finite size effects should be negligible since several Compton wavelengths fit within the total volume, giving a good approximation to the $\beta \to \infty$ limit.

All of our simulations are performed using hybrid Monte Carlo techniques. See [1] for details relating to the implementation for lattice SQM systems. Unlike [1], we do not use Fourier acceleration; rather, we have measured autocorrelation times and simply spaced our samples appropriately. Due to the simplicity of the fermion matrix, we were able to invert $M^T M s = \phi$ analytically, where ϕ is a pseudofermion configuration, using two steps of gaussian elimination. Alternatively, since $M^T M$ is a cyclic tridiagonal matrix, we were able to apply the Sherman-Morrison algorithms with partial pivoting. In either case, for the double precision that we used, we were able to keep the relative residual error $|(M^T M) \cdot s_{est} - \phi|/|\phi|$ of the inversion to less than 1×10^{-14} throughout the simulations. This is far more accurate than what was obtained using conjugate gradient. Finally, fit errors due to excited states were avoided by only fitting times t of Green functions $G(t)$ such that these effects could be shown to be negligible.

We have extracted excitation energies, or, effective masses, from connected Green functions $G^I(t)$ where $t = a, 2a, \ldots, Na$ is the imaginary-time of points on the lattice, and $I \in \{1B, 1F, 2B, 2F\}$ labels which excitation dominates the large-time behavior of the Green function. In particular, we choose

$$\begin{aligned} G^{1B}(t) &= \langle x_1 x_{1+t/a} \rangle\,, & G^{1F}(t) &= \langle \psi_1 \bar{\psi}_{1+t/a} \rangle\,, \\ G^{2B}(t) &= \langle x_1^2 x_{1+t/a}^2 \rangle_{conn.}\,, & G^{2F}(t) &= \langle x_1 \psi_1 x_{1+t/a} \bar{\psi}_{1+t/a} \rangle\,. \end{aligned} \tag{4.1}$$

Due to the symmetry $x \to -x$ of the action, as well as fermion number, the states that contribute to each of these Green functions come from different sectors of the state space. For $t \ll Na$ and $N \gg 1$, we have for example

$$G^{1B}(t) = c_{1B} e^{-m_{1B} t} + c_{3B} e^{-m_{3B} t} + \cdots\,, \qquad G^{2B}(t) = c_{2B} e^{-m_{2B} t} + c_{4B} e^{-m_{4B} t} + \cdots\,, \tag{4.2}$$

and similar equations for the fermions. Here $m_{1B} < m_{2B} < m_{3B} < m_{4B}$.

We now discuss how the contamination of higher excitations can be suppressed in (4.2), for the purpose of fitting the leading mode. In the limit of $g = 0$, we just have the simple harmonic oscillator with unit mass, frequency $\omega = m_{1B}$, as well as fermionic excitations that are degenerate with the bosons. For $g \neq 0$, we expect a spectrum similar to the anharmonic oscillator, so that $m_{3B} - m_{1B} > 2m_{1B}$ and similarly for the fermions. Thus the contribution of excited states to the Green function $G^{1B}(t)$ can be neglected if $t \gtrsim 2/m_{1B}$, since the relative exponential suppression in (4.2) will be something smaller than e^{-4}. To avoid finite size effects, which are of order $\exp[-(Na - t)m_{1B}]$, we choose $t \lesssim Na/3$. (Recall that $\beta = Na \equiv 1$ and $m_{1B} > 10$ in our simulations.) Taking these

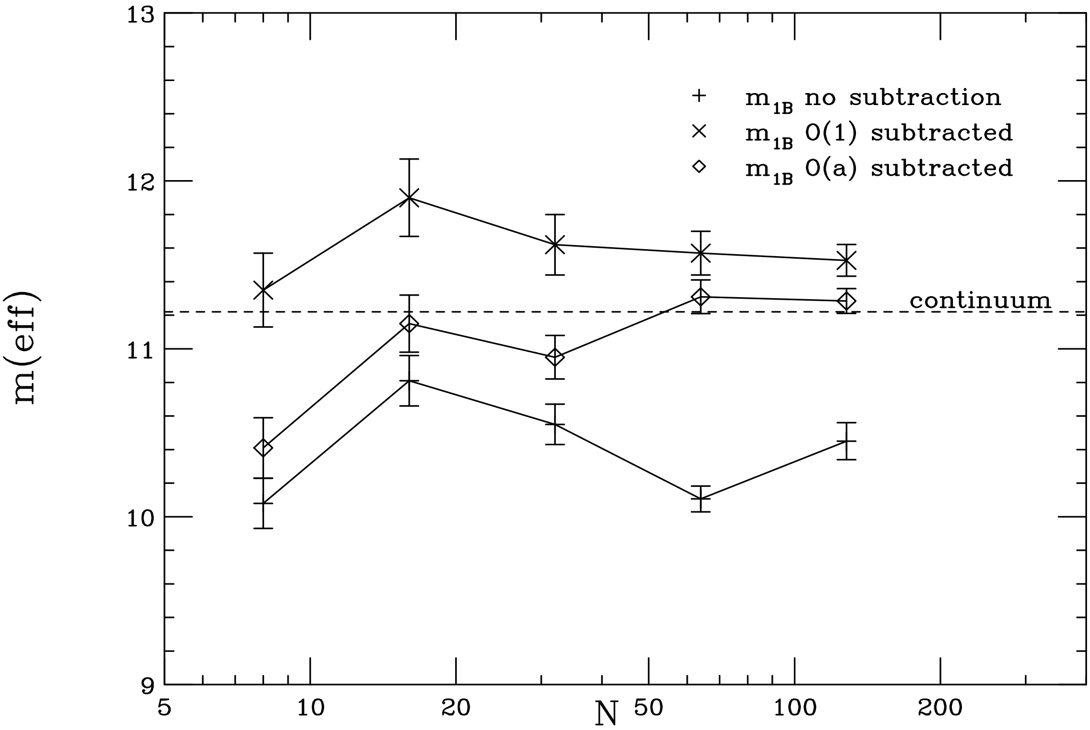

Figure 3: Leading boson mass for various forms of the naive action, with bare parameters $m = g = 10$. Large N corresponds to the continuum limit with $\beta = Na$ held fixed at $\beta = 1$. Lines are drawn to guide the eye.

constraints on t into account, we can safely linearize the fit to obtain the leading excitation for each Green function (i.e., fit $\ln G(t)$ to a linear function in t).

Three different versions of the naive action have been simulated: S, eq. (2.8), which has no corrections; S_c, eq. (3.8), which has the the $\mathcal{O}(1)$ counterterm; and S_{ca}, eq. (3.18), which includes the $\mathcal{O}(a)$ improvement.

As a first case, we take $g = 10$, where the perturbation expansion parameter $g/m^2 = 1/10$ is small. The results for m_{1B} are shown in figure 3, and for m_{1F} in figure 4. We compare these results to those of the continuum, obtained by a numerical solution of the Schrödinger equation corresponding to (2.1). It can be seen that the uncorrected naive action fails to approach the continuum limit for the boson. The discrepancy is of order 10 percent, consistent with an $\mathcal{O}(g/m^2)$ effect. The $\mathcal{O}(1)$-subtracted and $\mathcal{O}(a)$-improved results are in reasonable agreement with the continuum, though the latter is clearly better. The fermion mass does not appear to suffer from the same error, which is consistent with the fact that a counterterm is not required for it. However, it can be seen that the $\mathcal{O}(a)$-improved action S_{ca} more rapidly approaches the continuum result. (Recall that the correction to the fermion action occurs only at $\mathcal{O}(a)$.) In fact, at the largest N that we were able to achieve, S_{ca} obviously gives a superior approximation to the continuum results. We have also verified that the second set of excitations m_{2B} and m_{2F} approach the continuum results for the $\mathcal{O}(a)$-improved action S_{ca}. For brevity we do not present these results in detail.

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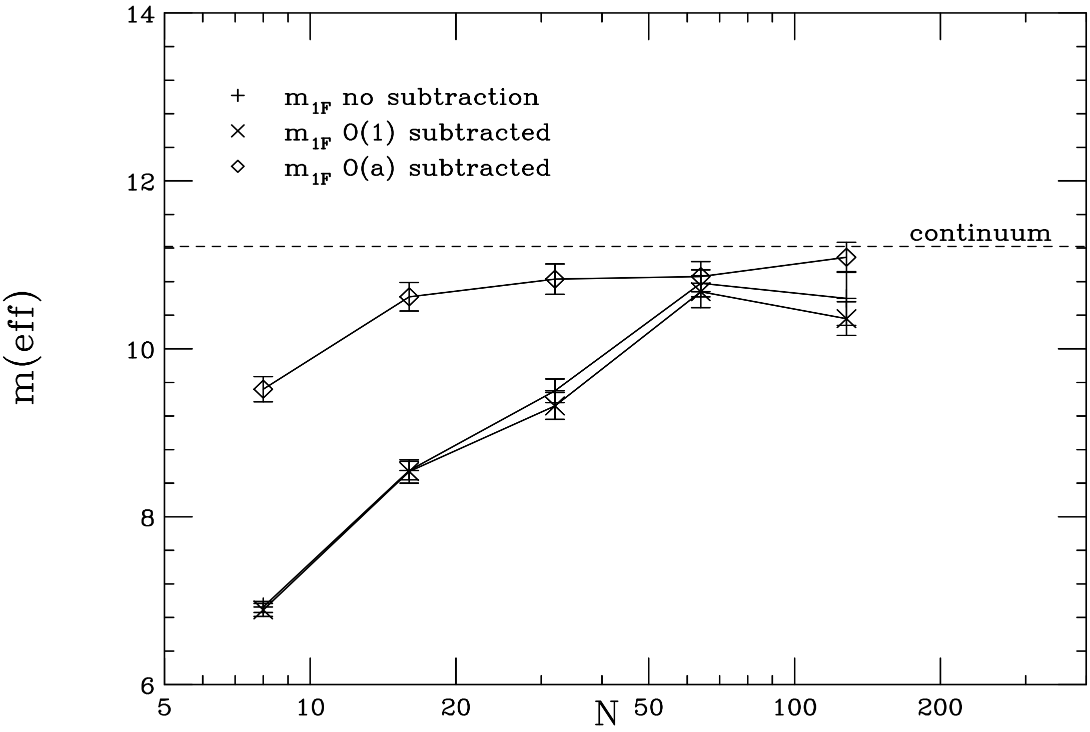

Figure 4: Similar to figure 3, except that the fermion mass is displayed.

Next, we have simulated the $\mathcal{O}(a)$-improved action at strong coupling, $g = 100$. The expansion parameter of perturbation theory would in this case be $g/m^2 = 1$, clearly beyond the range of validity of a perturbative approach. In figure 5 we give our results for the first boson and fermion excitations. It can be seen that, as indicated by the nonperturbative transfer matrix analysis of section 3.2, the desired continuum limit is obtained. For comparison, we show the results of CG [1], obtained using the susy lattice action (2.18). Although the $\mathcal{O}(a)$-improvement of the naive lattice action gives a superior approximation to the continuum, an $\mathcal{O}(a)$-improvement of the susy lattice action is straightforward to perform using the transfer matrix techniques of Section 3.2; this would also yield better agreement with the continuum.

5. Conclusions

In this article we have shown that a careful study of lattice power counting, or the transfer matrix, is necessary in order to understand the quantum continuum limit, even in the simple case of 1d SQM. The lower UV degrees that occur in lattice perturbation theory with some amount of exact susy provide a compelling indication of the following: one should make exact lattice susy a requirement of the construction to the extent that it is possible. As has been seen, in the case of the susy lattice action of SQM, the continuum limit is obtained without the need for counterterms, due to cancellations between diagrams with $D = 0$. The situation is similar in the field theory context (for example, compare to [3, appendix D]); although, in field theory we expect that some amount

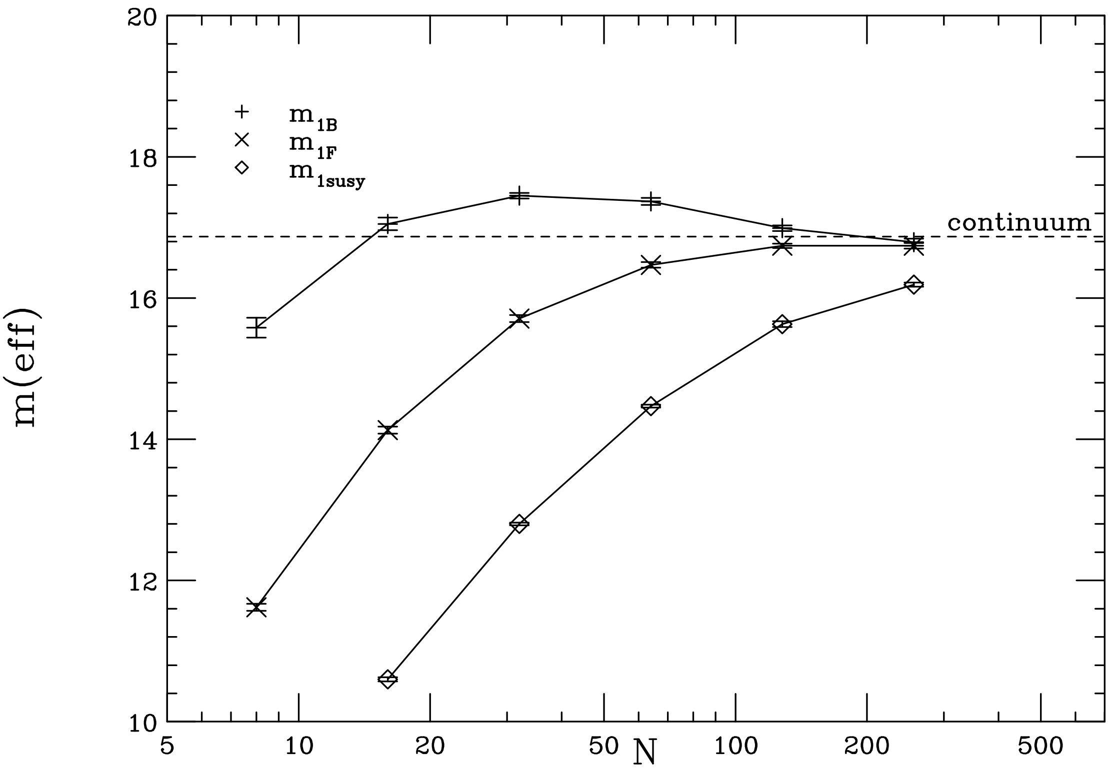

Figure 5: Leading boson and fermion mass for strong coupling, $g = 100, m = 10$, in the case of the $\mathcal{O}(a)$-improved naive action. It can be seen that a good approximation to the continuum results is obtained for reasonable N. For comparison, we also plot the susy lattice results reported in [1] (the boson and fermion masses are degenerate).

of fine-tuning of lattice counterterms will generically be required, particularly if the continuum field theory is not finite. This is because modes near the UV cutoff will not decouple in the $D \geq 0$ diagrams, yielding contributions that must be subtracted in order to match the continuum theory. Nevertheless, we expect the UV behavior of the lattice theory will be better if some exact susy is preserved. Practically speaking, this amounts to less counterterms that need to be calculated and adjusted. This is particularly important in a strong coupling regime, where the needed fine-tuning of lattice counterterms must be determined from a nonperturbative analysis such as Monte Carlo simulation.

It is also of interest to study these issues in models where symmetries other than susy are partly responsible for good UV behavior in the continuum theory. For example, certain classes of nonlinear σ-models are known to be renormalizable in the continuum because the form of counterterms is greatly restricted by the nonlinearly realized symmetry. It has been found that it is not possible to simultaneously preserve an exact lattice supersymmetry and the full nonlinear symmetry [3, 9]. Thus we expect the UV behavior of these models to be worse than in the continuum, and we are presently studying whether or not a finite set of counterterms suffices to guarantee the continuum limit.

Acknowledgments

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A. The transfer matrix of the naive lattice partition function

The purpose of this appendix is to show that eq. (3.12) is the transfer matrix of the partition function of the naive lattice action (2.8), with Wilson parameter $r = 1$ (see (2.12)), supplemented for further use below by the counterterms (3.11):

$$S[k,\ell] = \sum_{i=1}^{N} \left[\frac{1}{2a}(x_i - x_{i-1})^2 + \frac{1}{2a}(h_i' h_i' + k_i) + \bar{\psi}_i(\psi_i - \psi_{i-1}) + a(h_i'' + \ell_i)\bar{\psi}_i \psi_i \right], \quad \text{(A.1)}$$

where, as usual $k_i = k(x_i), \ell_i = \ell(x_i)$, with $i + N \equiv i$. The lattice partition function is then defined as:

$$Z[k,\ell] = c \prod_{i=1}^{N} \int d\bar{\psi}^i d\psi^i \int_{-\infty}^{\infty} dx^i \; e^{-S} \,. \quad \text{(A.2)}$$

To continue, it is convenient to change the fermionic variables as follows:

$$\psi_i = \bar{\eta}_{i+1}, \;\; \bar{\psi}_i = \eta_i \,, \quad \text{(A.3)}$$

so that the action (A.1) and partition function (A.2) become:

$$\begin{aligned} S[k,\ell] &= \sum_{i=1}^{N} \left[\frac{1}{2a}(x_i - x_{i-1})^2 + \frac{1}{2a}(h_i' h_i' + k_i) + \bar{\eta}_i \eta_i - \bar{\eta}_{i+1}\eta_i \left(1 + a(h_i'' + \ell_i)\right) \right], \\ Z[k,\ell] &= \tilde{c} \prod_{i=1}^{N} \int d\bar{\eta}^i d\eta^i \int_{-\infty}^{\infty} dx^i \; e^{-S} \,, \end{aligned} \quad \text{(A.4)}$$

where we absorbed a minus sign in the normalization constant $\tilde{c}$.

To construct the transfer matrix and hamiltonian (see, for example, [10]), we first introduce, at each time slice, a Hilbert space which is a tensor product of a bosonic and fermionic space. The bosonic Hilbert space is that of square integrable functions on the line. We use the basis of position eigenstates, $\{|x\rangle, \; \langle x'|x\rangle = \delta(x' - x)\}$, where the momentum and position operators, $[\hat{p}, \hat{q}] = -i$, act as $\hat{q}|x\rangle = |x\rangle x$ and $e^{i\hat{p}\Delta}|x\rangle = |x + \Delta\rangle$ (note that we continue using the dimensions of the previous section: x has mass dimension $-1/2$, a has dimension of length, while the superpotential $h(x)$ is dimensionless). The fermionic Hilbert space is two dimensional and is spanned by the vectors $|0\rangle$ and $|1\rangle$. The fermionic creation and annihilation operators obey $\{\hat{b}^\dagger, \hat{b}\} = 1$, such that $\hat{b}|0\rangle = 0$, $|1\rangle = \hat{b}^\dagger|0\rangle$. The fermionic coherent states are defined as $|\eta\rangle \equiv |0\rangle + |1\rangle\eta$, $\langle\eta| = \langle 0| + \bar{\eta}\langle 1|$, where η and $\bar{\eta}$ are Grassmann variables. We then recall the usual relations for the decomposition of unity, $\langle\eta'|\eta\rangle = e^{\bar{\eta}'\eta}$; $\hat{1} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}|\eta\rangle\langle\eta|$, and for traces of operators $\mathcal{O}$ on the fermionic Hilbert space: $\mathrm{Tr}\,\mathcal{O} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}\langle\eta|\mathcal{O}|-\eta\rangle$; $\mathrm{Tr}(-1)^F\mathcal{O} = \int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}\langle\eta|\mathcal{O}|\eta\rangle$, with $(-1)^F|0\rangle = |0\rangle$.

We then define the transfer matrix by the equality:

$$
\begin{aligned}
Z[k,\ell] &= \tilde{c}\prod_{i=1}^{N}\int d\bar\eta^i d\eta^i \int\limits_{-\infty}^{\infty} dx^i \; e^{-S} \equiv \mathrm{Tr}\,(-1)^F\;\hat{T}^N \\
&= \prod_{i=1}^{N}\int d\bar\eta^i d\eta^i e^{-\bar\eta^i\eta^i}\int\limits_{-\infty}^{\infty} dx^i \times \\
&\quad \times \langle \eta^N, x^N|\hat{T}|\eta^{N-1}, x^{N-1}\rangle\langle \eta^{N-1}, x^{N-1}|\hat{T}|\eta^{N-2}, x^{N-2}\rangle \times \cdots \\
&\quad \times \langle \eta^2, x^2|\hat{T}|\eta^1, x^1\rangle\langle \eta^1, x^1|\hat{T}|\eta^N, x^N\rangle\,, \qquad (A.5)
\end{aligned}
$$

or, equivalently, through its matrix elements, $\langle \eta^{i+1}, x^{i+1}|\hat{T}|\eta^i, x^i\rangle =$

$$
\tilde{c}^{1/N}\;\exp\left[-\frac{a}{2}\left(\frac{x^{i+1}-x^i}{a}\right)^2 - \frac{a}{2}((h'(x^{i+1})^2 + k(x^i)) + \bar\eta^{i+1}\eta^i\left(1 + a(h''(x^i) + \ell(x^i))\right)\right]. \qquad (A.6)
$$

(Substituting (A.6) into (A.5) one can immediately obtain (A.4)).

Now, we can use the fermion coherent state identity (following from the definitions given above) $\langle\eta'|1 - X\hat{b}^\dagger\hat{b}|\eta\rangle = e^{(1-X)\bar\eta'\eta}$, as well as the action of $\hat{p}, \hat{q}$ on the states $|x^i\rangle$, to verify that the transfer matrix operator $\hat{T}$, with matrix elements given by eq. (A.6), is:

$$
\hat{T} = \tilde{c}^{1/N}\int\limits_{-\infty}^{\infty} dz\;\exp\left(-\frac{z^2}{2a} - \frac{a}{2}(h'(\hat{q})^2 + k(\hat{q}))\right)\;\exp\left(iz\hat{p}\right)\;\left(1 + a(h''(\hat{q}) + \ell(\hat{q}))\;\hat{b}^\dagger\hat{b}\right), \qquad (A.7)
$$

as already stated in eq. (3.12) in the main text (where $\tilde{c}^{1/N}$ is called $\mathcal{N}(a)$).

We note that the properties of the lattice partition function and transfer matrix of the supersymmetric lattice action are studied in [3].

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